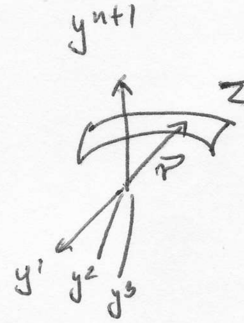
SB 11.18.2008
SB. 11. 25. 2008

Hypersurfaces isometrically immersed in $\mathbb{R}^{n+1}$.



$$g = dy^{1^2} + dy^{2^2} + \dots + (dy^{n+1})^2 = (d\vec{r})^2$$

$$d\vec{r} = (dy^1, dy^2, \dots, dy^{n+1})$$

$$\Sigma = \{ \vec{r} = (y^1, \dots, y^{n+1}) \in \mathbb{R}^{n+1} : \ \vec{r} = \vec{r}(x^1, \dots x^n) \quad dx^1 \wedge \dots dx^n \neq 0 \}.$$

$$g|_\Sigma = |d\vec{r}(x^1, \dots x^n)|^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \theta^\mu \theta^\nu = \underbrace{\theta^{1^2} + \dots + \theta^{n^2}}_{\text{orthonormal frame}}$$

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$(\theta^1, \dots, \theta^n)$ orthonormal frame on Σ

$(e_1, \dots e_n)$ dual frame on Σ.

We tautologically have:

(2) $\boxed{d\vec{r} = e_1(\vec{r})\theta^1 + e_2(\vec{r})\theta^2 + \dots + e_n(\vec{r})\theta^n}$ (e.g. $e_3 \lrcorner d\vec{r} = e_3(\vec{r})$)

and let us denote $e_\mu(\vec{r})$ by $\vec{e}_\mu$:

$\boxed{\vec{e}_\mu = e_\mu(\vec{r}).}$ — Vectors in $\mathbb{R}^{n+1}$!!!

Fact

$\boxed{\vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu}.}$ (1)

Proof

$$\theta^{1^2} + \dots + \theta^{n^2} = \delta_{\mu\nu}\theta^\mu\theta^\nu = (d\vec{r})^2 = (\vec{e}_\mu \theta^\mu)\cdot(\vec{e}_\nu \theta^\nu) = (\vec{e}_\mu \cdot \vec{e}_\nu)\theta^\mu\theta^\nu$$

$\square$.

Let $\vec{n} \in \mathbb{R}^{n+1}$ s.t. $\forall (x^1, \dots x^n) = \vec{r}$ $\quad \vec{n}\cdot\vec{e}_\mu = 0$ and $\vec{n}^2 = 1$.

Thus

$(\vec{e}_\mu, \vec{n})$ is an orthonormal basis in $\mathbb{R}^{n+1}$ at each point of Σ!.

Cartan's lemma

Let $A_{\mu\nu}$, $\mu,\nu=1,\dots n$ be 1-forms on n-dimensional manifold M s.t. $A_{\mu\nu}=-A_{\nu\mu}$, Let θ^μ be a frame on M.

If $\quad A_{\mu\nu}\wedge\theta^\nu=0 \quad$ then $\quad A_{\mu\nu}=0.$

Proof

$$0=A_{\mu\nu}\wedge\theta^\nu = A_{\mu\nu\rho}\theta^\rho\wedge\theta^\nu=0 \Rightarrow A_{\mu[\nu\rho]}=0 \Rightarrow$$

$$\begin{cases} \cancel{A_{\mu\nu\rho}} - A_{\mu\rho\nu}=0 \\ A_{\rho\mu\nu} - \cancel{A_{\rho\nu\mu}}=0 \\ -\ \cancel{A_{\nu\rho\mu}} - \cancel{A_{\nu\mu\rho}}=0 \end{cases} \Leftarrow \text{ but; } A_{\mu\nu\rho}=-A_{\nu\mu\rho}$$

$$2A_{\rho\mu\nu}=0 \Rightarrow A_{\rho\mu\nu}\theta^\nu = A_{\rho\mu}=0. \quad \square.$$

How the Levi-Civita connection $\Gamma_{\mu\nu}$ of the induced metric $g=\delta_{\mu\nu}\theta^\mu\theta^\nu$ look like?

Calculate $d\vec{e}_\mu$:

$$\boxed{d\vec{e}_\mu = b_\mu\vec{n} + \gamma_{\nu\mu}\vec{e}_\nu}$$

This defines 1-forms b_μ and $\gamma_{\mu\nu}$ on Σ.

Obviously

$$\boxed{\begin{cases} b_\mu = d\vec{e}_\mu\cdot\vec{n} \\ \gamma_{\nu\mu} = d\vec{e}_\mu\cdot\vec{e}_\nu \end{cases}}$$

and are totally determined by specifying $\vec{r}=\vec{r}(x^\mu)$ defining Σ.

Fact

$\gamma_{\mu\nu}=-\gamma_{\nu\mu}.$

Proof

$$\delta_{\mu\nu}=\vec{e}_\mu\cdot\vec{e}_\nu \Rightarrow 0=d\vec{e}_\mu\cdot\vec{e}_\nu+\vec{e}_\mu\cdot d\vec{e}_\nu = \gamma_{\mu\nu}+\gamma_{\nu\mu} \quad \square.$$

Of course $d\theta^\mu$ is decomposable onto $\theta^\nu \wedge \theta^\sigma$ and one can look for $\Gamma_{\mu\nu}$ s.t. $d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0$, $\Gamma_{\mu\nu} = \delta_{\mu\rho}\Gamma^\rho{}_\nu$, $\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0$.

We have:

<u>Proposition</u>

1) $\Gamma_{\mu\nu} = \gamma_{\mu\nu} = d\vec{e}_\mu \cdot \vec{e}_\nu$

2) $b_{\mu\nu} = b_{\nu\mu}$ where $b_\mu = d\vec{e}_\mu \cdot \vec{n} = b_{\mu\nu}\theta^\nu$.

<u>Proof</u>

We use (2) on Σ:

$$0 = d^2\vec{r} = d(\vec{e}_\mu \theta^\mu) = d\vec{e}_\mu \wedge \theta^\mu + \vec{e}_\mu d\theta^\mu =$$

$$= d\vec{e}_\mu \wedge \theta^\mu - \vec{e}_\mu \Gamma^\mu{}_\nu \wedge \theta^\nu = (b_\mu \vec{n} + \gamma_{\nu\mu}\vec{e}_\nu) \wedge \theta^\mu - \vec{e}_\nu \Gamma^\nu{}_\mu \wedge \theta^\mu =$$

$$= b_\mu \wedge \theta^\mu \vec{n} + (\gamma_{\nu\mu} - \Gamma^\nu{}_\mu) \wedge \theta^\mu \vec{e}_\nu =$$

$$= b_\mu \wedge \theta^\mu \cdot \vec{n} + (\gamma_{\nu\mu} - \Gamma_{\nu\mu}) \wedge \theta^\mu \vec{e}_\nu$$

$$\Rightarrow \begin{cases} b_\mu \wedge \theta^\mu = 0 \\ \underbrace{(\gamma_{\nu\mu} - \Gamma_{\nu\mu})}_{\text{antisymmetric + Cartan's lemma}} \wedge \theta^\mu = 0 \end{cases} \quad \Rightarrow b_{\mu\nu}\theta^\nu \wedge \theta^\mu = 0 \Rightarrow b_{[\mu\nu]} = 0$$

$$\Downarrow$$

$b_{\mu\nu}$ is symetric.

$$\Rightarrow \Gamma_{\nu\mu} = \gamma_{\nu\mu}$$

$\square$.

Def

Form $b = b_{\mu\nu}\,\theta^\mu \cdot \theta^\nu$ where $b_\mu = d\vec{e}_\mu \cdot \vec{n} = b_{\mu\nu}\theta^\nu$ is called 2nd fundamental form for Σ.

We have

$$\boxed{d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu} \qquad (3)$$

where $\Gamma_{\nu\mu} = d\vec{e}_\mu \cdot \vec{e}_\nu$ are the Levi-Civita connection 1-forms for $g|_\Sigma = \delta_{\mu\nu}\,\theta^\mu\theta^\nu$.

Proposition

$$\boxed{d\vec{n} = -b_\mu \vec{e}_\mu} \qquad (4)$$

Proof

$\vec{n}^2 = 1 \Rightarrow d\vec{n}\cdot\vec{n} = 0$ (*)

$\vec{e}_\mu \cdot \vec{n} = 0 \Rightarrow \underbrace{d\vec{e}_\mu \cdot \vec{n}}_{=\,b_\mu} = -\vec{e}_\mu \cdot d\vec{n}$ (**)

$\Rightarrow \quad d\vec{n} = \alpha\vec{n} + \beta_\mu \vec{e}_\mu$ and $\alpha = 0$

(**) $\beta_\mu = -b_\mu$ $\square$.

Compatibility conditions for

$$\begin{cases} d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n} = -b_\mu \vec{e}_\mu \end{cases}$$

$$d^2\vec{e}_\mu = 0 = db_\mu \vec{n} + b_\mu\wedge b_\varrho \vec{e}_\varrho + d\Gamma_{\nu\mu}\vec{e}_\nu - \Gamma_{\nu\mu}\wedge(b_\nu\vec{n} + \Gamma_{\varrho\nu}\vec{e}_\varrho)$$

$$= (db_\mu + b_\nu\wedge\Gamma_{\nu\mu})\vec{n} + (d\Gamma_{\nu\mu} - \Gamma_{\varrho\mu}\wedge\Gamma_{\nu\varrho} + b_\mu\wedge b_\nu)\vec{e}_\nu$$

$$\Rightarrow \quad \boxed{\begin{array}{l} db_\mu + b_\nu\wedge\Gamma_{\nu\mu} = 0 \\ \Omega_{\nu\mu} = b_\nu\wedge b_\mu \end{array}} \quad \begin{array}{l} \leftarrow \text{Codazzi} \\ \leftarrow \text{Gauss} \end{array}$$

The second compatibility conditions:

$$d^2\vec{n} = 0 = -db_\mu\vec{e}_\mu + b_\mu\wedge(b_\mu\vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu) =$$

$$= (-db_\mu + b_\nu\wedge\Gamma_{\mu\nu})\vec{e}_\mu + b_\mu\wedge b_\mu\vec{n}$$ satisfied iff G-Codazzi satisfied.

Thm

Let (M,g) be an n-dimensional Riemannian manifold. Let b be a bilinear symmetric form on M and let (θ^μ) be an orthonormal coframe for g, $g = \delta_{\mu\nu}\theta^\mu\theta^\nu$. Then b can be a second fundamental form for an isometric immersion of g in $\mathbb{R}^{n+1}$ with the standard euclidean metric provided that

$$(G\text{-}C)\quad \begin{cases} db_\mu + b_\nu\wedge\Gamma_{\nu\mu} = 0 \\ \Omega_{\nu\mu} = b_\nu\wedge b_\mu \end{cases}$$

where $\Gamma_{\mu\nu}$ and $\Omega_{\nu\mu}$ are respective L.C. connection 1-forms and curvature 2-forms in coframe (θ^μ) and b_ν is defined by $\quad b = b_{\mu\nu}\theta^\mu\theta^\nu, \quad b_\nu = b_{\nu\mu}\theta^\mu.$

Rmk. If $n=2$ conditions (G-C) are also sufficient.

If the Gauss-Codazzi equations are satisfied the equations to be solved to get the isometric immersion are:

$$\begin{cases} d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n} = -b_\mu \vec{e}_\mu \end{cases}$$

Example

Take $\boxed{g = dx^2 + 2\cos\omega\, dx\, dy + dy^2}$, (M) $\qquad \omega = \omega(x,y)$

Curvature:

$$g = (dx + \cos\omega\, dy)^2 - \cos^2\omega\, dy^2 + dy^2 = (dx + \cos\omega\, dy)^2 + \sin^2\omega\, dy^2$$

$$\begin{cases} \theta^1 = dx + \cos\omega\, dy \\ \theta^2 = \sin\omega\, dy \end{cases} \qquad g = \theta^{1\,2} + \theta^{2\,2}$$

$$\boxed{\begin{aligned} d\theta^1 &= -\omega_x \sin\omega\, dx\, dy = -\omega_x\, dx\, \theta^2 = -\omega_x \theta^1\theta^2 \\ d\theta^2 &= \omega_x \cos\omega\, dx\, dy = \omega_x \cos\omega\, \theta^1 \frac{1}{\sin\omega}\theta^2 = \omega_x \cot\omega\, \theta^1\theta^2 \end{aligned}}$$

$$d\theta^1 = -\Gamma^1{}_2\theta^2 = -\Gamma_{12}\theta^2 = -\omega_x \theta^1\theta^2$$

$$\Rightarrow \Gamma_{12} = \omega_x \theta^1 + \alpha\theta^2$$

$$d\theta^2 = -\Gamma^2{}_1\theta^1 = \Gamma_{12}\theta^1 = \alpha\theta^2\theta^1 = \omega_x \cot\omega\, \theta^1\theta^2$$

$$\Rightarrow \boxed{\Gamma_{12} = \omega_x(\theta^1 - \cot\omega\, \theta^2)} = \omega_x\, dx$$

$$\Omega_{12} = d\Gamma_{12} + 0 = d(\omega_x dx) = \omega_{xy}\, dy\, dx = -\frac{\omega_{xy}}{\sin\omega}\theta^1 \wedge \theta^2$$

Proposition

Metric (M) has curvature K if

$$\boxed{\omega_{xy} = -K\sin\omega} \qquad K = K(x,y).$$

In particular: (M) has scalar curvature equal to

$\boxed{K=-1}$ ASSUMED FROM NOW, ON

iff function ω satisfies 'sine-Gordon' equation!

$$\boxed{\omega_{xy} = \sin\omega}$$

When g as in (M) can be isometrically immersed in $\mathbb{R}^3$ with $b = 2b_{xy}\,dx\,dy$ being the second fundamental form?

$$b = 2b_{xy}(\theta^1 - \cot\omega\,\theta^2)\frac{1}{\sin\omega}\theta^2$$

$$b_1 = \frac{b_{xy}}{\sin\omega}\theta^2$$

$$b_2 = \frac{b_{xy}}{\sin\omega}\theta^1 - 2\frac{b_{xy}}{\sin\omega}\cot\omega\,\theta^2$$

Gauss equation:

$$-\theta^1\wedge\theta^2 = \Omega_{12} = b_1\wedge b_2 = -\frac{b_{xy}^2}{\sin^2\omega}\theta^1\wedge\theta^2$$

$$\Rightarrow \quad \boxed{b_{xy} = \sin\omega}$$

Thus $\boxed{b = 2\sin\omega\,dx\,dy}$

Codazzi equations: $\quad b_1 = \theta^2, \quad b_2 = \theta^1 - 2\cot\omega\,\theta^2$

$$db_1 = \omega_x\cot\omega\,\theta^1\theta^2$$

C.E.1 $\quad \| $

$$-b_2\wedge\Gamma_{21} = b_2\Gamma_{12} = (\theta^1 - 2\cot\omega\,\theta^2)\,\omega_x(\theta^1 - \cot\omega\,\theta^2) = -\omega_x\cot\omega\,\theta^2\theta^1 \quad ✓$$

C.E.$_2$ $\quad d\theta_2 = -\omega_x \theta^1\theta^2 + 2\frac{\omega_x}{\sin^2\omega}\theta^1\theta^2 - 2\cot^2\omega\, \omega_x \theta^1\theta^2 =$

$$= \omega_x \frac{-\sin^2\omega + 2 - 2\cos^2\omega}{\sin^2\omega}\theta^1\theta^2 = \omega_x\theta^1\theta^2$$

$\parallel$

$-b_1\Gamma_{12} = -\theta^2\omega_x\theta^1$ $\quad$ ✓

Thus if

$$\boxed{\begin{array}{ll} g = dx^2 + 2\cos\omega\, dx\,dy + dy^2 & \\ b = 2\sin\omega\, dx\,dy & \omega_{xy} = \sin\omega \end{array}}$$

then g can be isometrically immersed in $\mathbb{R}^3$ with b being the II^{nd} fundamental form.

<u>Examples</u> of solutions to

$$\begin{cases} d\vec{e_\mu} = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e_\nu} \\ d\vec{n} = -b_\mu \vec{e_\mu} \end{cases}$$

<u>Homework</u>

Show that the surface in $\mathbb{R}^3$

$$z = -\sqrt{1-x^2-y^2} + \log\frac{1+\sqrt{1-x^2-y^2}}{\sqrt{x^2+y^2}}$$

correspond to a solution of the sine-gordon equation

$$\omega_{xy} = \sin\omega,$$

written in coordinates $x = \frac{\tau+\xi}{\sqrt{2}}$, $y = \frac{\tau-\xi}{\sqrt{2}}$,

not depending on ξ, and vanishing when $\tau\to\infty$.

Immersing n-dimensional manifolds in $\mathbb{R}^{n+k}$.

$$g = dy^{1\,2} + \dots + dy^{n+k\,2} = (d\vec{r})^2$$

$$\Sigma_n = \{ \vec{r} = (y^1, \dots, y^{n+k}) \in \mathbb{R}^{n+k}, \quad \vec{r} = \vec{r}(x^1, \dots x^n)$$
$$dx^1 \wedge \dots \wedge dx^n \neq 0 \ \}.$$

$$g|_\Sigma = |d\vec{r}|^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \delta_{\mu\nu} \theta^\mu \theta^\nu = \theta^{1\,2} + \dots + \theta^{n\,2}$$

$\{\theta^\mu\}$ orthonormal coframe, $\quad \mu = 1, \dots n$
$\{e_\mu\}$ dual frame

$$d\vec{r} = e_\mu(\vec{r})\theta^\mu \qquad e_\mu(\vec{r}) = \vec{e}_\mu$$

$\vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu}$ - orthonormal vectors in $\mathbb{R}^{n+k}$

Let $(\vec{n}_a)$ $a = 1, \dots k$ be vectors in $\mathbb{R}^{n+k}$ which are

- normal to Σ at each point of Σ $\quad \vec{n}_a \cdot \vec{e}_\mu = 0$
- orthogonal to each other $\quad \vec{n}_a \cdot \vec{n}_b = 0 \quad a \neq b$
- unital $\quad \vec{n}_a^{\,2} = 1 \quad \forall a = 1, \dots k$

$\Big\}\ \vec{n}_a \cdot \vec{n}_b = \delta_{ab}$

$(\vec{e}_\mu, \vec{n}_a)$ is orthonormal basis in $\mathbb{R}^{n+1}$ attached to each point of Σ!

$$\begin{cases} d\vec{e}_\mu = b_{\mu a} \vec{n}_a + \gamma_{\nu\mu} \vec{e}_\nu \\ d\vec{n}_a = \alpha_{ab} \vec{n}_b + \beta_{\mu a} \vec{e}_\mu \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_{\nu\mu} = d\vec{e}_\mu \cdot \vec{e}_\nu \\ b_{\mu a} = d\vec{e}_\mu \cdot \vec{n}_a \\ \alpha_{ab} = d\vec{n}_a \cdot \vec{n}_b \\ \beta_{\mu a} = d\vec{n}_a \cdot \vec{e}_\mu \end{cases}$$

$$0 = d(\vec{e}_\mu \cdot \vec{e}_\nu) = \gamma_{\nu\mu} + \gamma_{\mu\nu} \quad \Rightarrow \quad \gamma_{\mu\nu} = -\gamma_{\nu\mu}$$

$$0 = d^2\vec{r} = d\vec{e}_\mu \theta^\mu + \vec{e}_\mu \, d\theta^\mu =$$

$$= \left(b_{\mu a}\vec{n}_a + \gamma_{\nu\mu}\vec{e}_\nu\right)\theta^\mu + \vec{e}_\mu \left(-\Gamma_{\mu\rho}\theta^\rho\right) =$$

$$= b_{\mu a}\theta^\mu \vec{n}_a + \left(\gamma_{\nu\mu} - \Gamma_{\nu\mu}\right)\wedge\theta^\mu \vec{e}_\nu$$

cartan's lemma.

$$\Rightarrow \quad b_{\mu a}\theta^\mu = 0 \quad \text{and} \quad \boxed{\gamma_{\nu\mu} = \Gamma_{\nu\mu}}$$

$$b_{\mu\rho a}\,\theta^\rho \wedge \theta^\mu = 0 \qquad\qquad b_{\mu a} = b_{\mu\rho a}\theta^\rho$$

$$\Rightarrow \quad \boxed{b_{\mu\rho a} = b_{\rho\mu a}}$$

$$\boxed{b_a = b_{\mu\rho a}\theta^\mu\theta^\rho}$$ ← second fundamental form along $\vec{n}_a$.

$$\boxed{d\vec{e}_\mu = b_{\mu a}\vec{n}_a + \Gamma_{\nu\mu}\vec{e}_\nu}$$

$b_{\mu a} \rightsquigarrow b_a$

↑ second fundamental form.

$$\vec{n}_a \cdot \vec{n}_b = \delta_{ab} \Rightarrow d\vec{n}_a \cdot \vec{n}_b + \vec{n}_a \cdot d\vec{n}_b = 0$$

$$\boxed{\alpha_{ab} = -\alpha_{ba}}$$

$$0 = d(\vec{n}_a \cdot \vec{e}_\mu) = d\vec{n}_a \vec{e}_\mu + \vec{n}_a d\vec{e}_\mu = b_{\mu a} + d\vec{n}_a \vec{e}_\mu = b_{\mu a} + \beta_{\mu a}$$

$$\beta_{\mu a} = -b_{\mu a}$$

$\Rightarrow$

$$\begin{cases} d\vec{e}_\mu = b_{\mu a}\vec{n}_a + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n}_a = \alpha_{ab}\vec{n}_b - b_{\mu a}\vec{e}_\mu \end{cases} \qquad \begin{cases} b_{\mu a} = d\vec{e}_\mu \cdot \vec{n}_a \\ \alpha_{ab} = d\vec{n}_a \cdot \vec{n}_b \end{cases}$$

$$\alpha_{ab} = -\alpha_{ba}$$

$$b_{\mu\nu a} = b_{\nu\mu a} \text{ where } b_{\mu a} = b_{\mu\nu a}\theta^\nu$$

Compatibility:

$$0 = \underline{db_{\mu a}\vec{n}_a} + b_{\mu a}(\underline{\alpha_{ab}\vec{n}_b} - b_{\mu a}\vec{e}_\mu) +$$
$$+ d\Gamma_{\nu\mu}\vec{e}_\nu - \Gamma_{\nu\mu}(b_{\underline{\nu a}}\vec{n}_a + \Gamma_{\rho\nu}\vec{e}_\rho) =$$
$$= (db_{\mu a} - b_{\mu b}\alpha_{ba} + b_{\nu a}\Gamma_{\nu\mu})\vec{n}_a +$$
$$+ (d\Gamma_{\nu\mu} - \Gamma_{\rho\mu}\Gamma_{\nu\rho} + b_{\mu a}b_{\nu a})\vec{e}_\nu$$

$\Rightarrow$

$$\boxed{\begin{array}{ll} db_{\mu a} + b_{\nu a}\Gamma_{\nu\mu} - b_{\mu b}\alpha_{ba} = 0 & \leftarrow \text{Codazzi} \\ \Omega_{\nu\mu} = b_{\nu a}\wedge b_{\mu a} & \leftarrow \text{Gauss} \end{array}}$$

$$0 = \underline{d\alpha_{ab}\vec{n}_b} + \alpha_{ab}d\vec{n}_b - db_{\mu a}\vec{e}_\mu + b_{\mu a}d\vec{e}_\mu =$$
$$= d\alpha_{ab}\vec{n}_b - \alpha_{ab}(\alpha_{bc}\vec{n}_c - b_{\mu b}\vec{e}_\mu) +$$
$$+ (b_{\nu a}\Gamma_{\nu\mu} - b_{\mu b}\alpha_{ba})\vec{e}_\mu - b_{\mu a}(b_{\mu b}\vec{n}_b + \Gamma_{\nu\mu}\vec{e}_\nu)$$

$$d\alpha_{ab} - \alpha_{ac} \wedge \alpha_{cb} - b_{\mu a} \wedge b_{\mu b} = 0$$

$$\cancel{\alpha_{ab} b_{\mu b}} + \cancel{b_{\nu a} \Gamma_{\nu\mu}} - \cancel{b_{\mu b} \alpha_{ba}} - \cancel{b_{\nu a} \Gamma_{\mu\nu}} = 0$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\Omega_{\nu\mu} = b_{\nu a} \wedge b_{\mu a} && \longleftarrow \text{Gauss} \\ &d b_{\mu a} + b_{\nu a} \wedge \Gamma_{\nu\mu} - b_{\mu b} \wedge \alpha_{ba} = 0 && \longleftarrow \text{Codazzi} \\ &d\alpha_{ab} - \alpha_{ac} \wedge \alpha_{cb} = b_{\mu a} \wedge b_{\mu b} && \longleftarrow \text{Ricci} \end{aligned}}$$

α_{ab} are (sometimes) called torsions

<u>Thm</u> Schlaffli (Cartan).

Any analytic Riemannian manifold of dimmension n can be locally isometrically embedded in a Riemannian ~~man~~ <u>flat</u> manifold of dimension $N \leq \frac{n(n+1)}{2}$
↑
If <u>flat</u> is replaced by Ricci flat then $N \leq n+1$

(Campbell 1926
A course of differential geometry
Clarendon Press Oxford)